Polyminos : a way to teach the mathematical concept of implication

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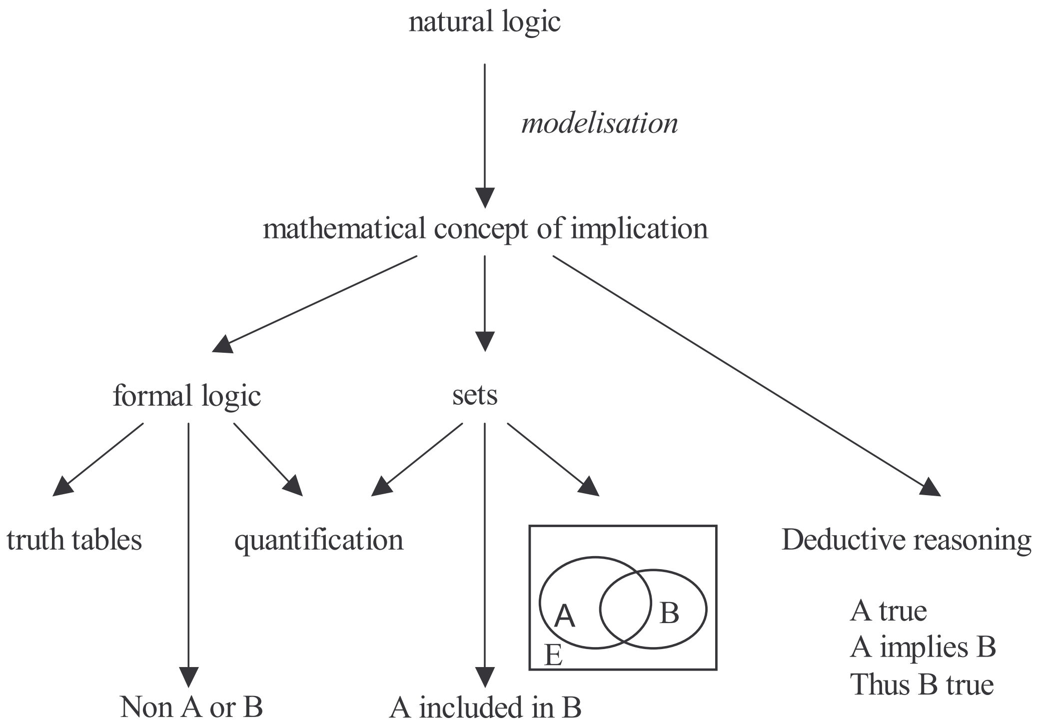
Abstract : In this paper, we present a didactic analysis of the implication under three points of view: sets, formal logic, deductive reasoning. For this study, our hypothesis is that most of the difficulties on implication are due to the lack of links in education between those three points of view. Then, we show, thanks to the analysis of a problem from our experimentations, how polyminos' paving can imply the sets point of view.

INTRODUCTION

The existence of the implication as an object of our everyday life leads to confuse it with the mathematical object. As a result, the implication seems to be a clear object. Yet, students have difficulties related to this concept until the end of university, especially with regard to necessary and sufficient conditions.
The study we present here is a part of our thesis on the mathematical concept of implication. Our theorical framework is placed in the theory of french didactics, in particular, we use the tools of Vergnaud's conceptuals fields theory and those of Brousseau's didactical situations theory. Our study is based on the work of V. Durand-Guerrier [Durand-Guerrier, 1999] on the one hand and of J. Rolland [Rolland, 1998] on the other hand. V. Durand-Guerrier shows, in particular, the importance of the contingent statements for the comprehension of the implication. J. Rolland is interested in the distinction between sufficient and necessary condition. Our researches are also linked to those of J. and M. Rogalski. They try to define types of structuring of the use of logic when evaluating the truth of an implication with a false premise.
We present three points of view on the implication, then the analysis of a problem of paving that allows a work on implication, we conclude with some results.

THREE POINTS OF VIEW ON THE IMPLICATION

The mathematical implication seems to be a model of the natural logic implication. Like any model, this mathematical concept is faithful from certain angles to that of natural logic but not from others. This distance between the mathematical concept and the object of our everyday life leads to obstacles in the use of the mathematical concept. An epistemological analysis [Deloustal, 2000] enabled us to distinguish three points of view on the implication : formal logic point of view, deductive reasoning point of view, sets point of view.

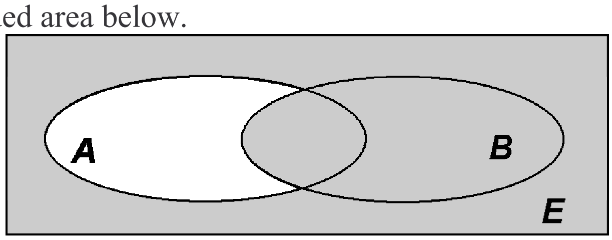


Of course, these three points of view are linked and their intersections are not empty. We will not develop here the formal logic point of view (for example truth tables or formal writing of the implication).

We call "deductive reasoning" the structure of an inference step : "A is true ; A implies B is true ; Thus B is true". Its ternary structure includes a premise "A is true", the reference to an established knowledge "A $\Rightarrow$B" (theorem, property or definition...) and a conclusion "B is true" [Duval, 1993, p 44]. In the deductive reasoning, the implication is used only as a tool. However, in French secondary education, where this point of view is the only one, it often acts as a definition for the implication.

Generally speaking, having a sets point of view, means to consider that properties define sets of objects : to each property corresponds a set, the set of the objects which satisfy this property.

The sets point of view on the implication can then be expressed as follows : in the set ***E***, if ***A*** (resp. ***B***) is the set of objects satisfying the property A (resp. B), then, the implication A $\Rightarrow$ B is satisfied by all the objects of the set ***E*** excluded those which are in ***A*** without being in ***B***, i.e. by all the objects located in the shaded area below.



RESEARCH HYPOTHESIS

The experiments carried out for three years, within the framework of our research, have shown that the implication was not a clear object even for beginner teachers. Moreover, they showed that, contrary to a widespread idea, a logic lecture is not enough to get rid of these mistakes and difficulties.

Following these comments, we formulate the research hypothesis : it is necessary to know and establish links between these three points of view on the implication for a good apprehension and a correct use of it.

In the following paragraph we show that a problem of paving, using only easy properties, may question the reasoning in a non obvious way and allow a work on the implication under the sets point of view.

THE EXPERIMENTATION

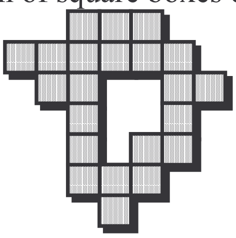
The problems we present here result from an experimentation carried out in 2001 with beginner teachers of mathematics. This experimentation includes two sessions on the proof and, in particular, on the implication. The first one contained two problems (one in geometry, the other one on pavings) the second one proposed a work on written proofs. For each meeting, a work by small groups was following an individual work to allow questionings and discussions.We present here two problems concerning pavings of polyminos.

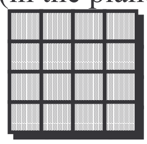
FIRST PROBLEM

Presentation

Here is the problem such as we gave it to the students. We will not give to the reader definitions of polyminos but those we gave to the students.

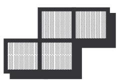
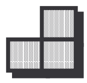
Définitions
Polymino : collection of square boxes connected by ridges (in the plane)



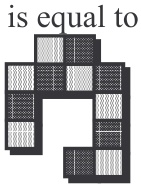


Domino : two-boxes polymino 

Size of a polymino : number of boxes
Even : a polymino is even if its size is even
Paving a polymino with dominos : to entirely cover, without overlap, a polymino with dominos. In this case, we say that a polymino is pavable with dominos.

pavable  non pavable 

Balanced : If one colours the polymino like a chessboard, we say that it is balanced when the number of white boxes is equal to the number of black boxes.



Question
Let P1, P2 and P3 be the properties of a polymino :

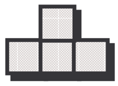
(P1) : pavable (by dominos)
(P2) : balanced
(P3) : even

Which are the relations between these properties?
Write a proof satisfying all the group.

Mathematical Analysis

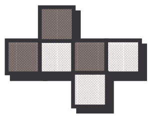
The implication P1⇒P3 is true because a pavable polymino can be paved with dominos, and dominos include two boxes. So the number of boxes included in a pavable polymino, that is to say its size, is an even number.

The implication P3⇒P1 is wrong, see counter-example 1:



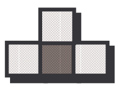
The implication P1⇒P2 is true. Indeed, let us suppose the polymino coloured as a chessboard, then a domino cover exactly one white box and one black box. Moreover, a pavable polymino can be covered by dominos. Conséquently, a pavable polymino includes the same number of white or black boxes, i.e. it is balanced.

The implication P2⇒P1 is wrong, see counter-example 2:



The implication P2⇒P3 is right. Since the polymino is balanced it contains the same number of black boxes and white boxes. Let n be this number. Then, the polymino's size is 2n and the polymino is even.

The implication P3⇒P2 is wrong, as shows the counter-example 1:



SECOND PROBLEM

Presentation

1 – We examine the implication P1⇒P2.
Here is a proof suggested by some and refused by others:

> A domino covers a white box and a black one. If a polymino is pavable, it is covered by k dominos and these k dominos cover k white boxes and k black boxes, therefore it is balanced.

Give your opinion on this proof.
Would you have a different answer as a mathematician and as a teacher?

2- We examine the implication P2⇒P1.
Here is a suggested proof:

> In a 2n-boxes balanced polymino, there are two neighbour boxes: the colours of these two boxes are different and the two boxes form a domino. Let us remove it. We obtain, then, a 2(n-1)-boxes balanced polymino. When repeating this process, we obtain a two-boxes balanced polymino. This shows that the polymino was pavable.

Give your opinion on this proof. Justify.

Mathematical Analysis

The implication P1⇒P2 (pavable ⇒ balanced) is true and the proposed proof is valid.
The implication P2⇒P1 (balanced ⇒ pavable) is false as shows the counter-example 2. Moreover, the proof is false since in a polymino boxes must be connected by a ridge. However, when one removes a domino taken randomly, this connexity is not assured and one obtains not inevitably a polymino. In addition, it is not always possible to choose the polymino that one removes so that the connexity is assured as shows the counter-example 2. It is balanced but whatever the removed domino, one cannot obtain a polymino.

DIDACTICAL ANALYSIS

General choices

Mathematical framework for the problem
First of all, we choose, for our experimentations, very easily accessible mathematical concepts. Indeed, our hypothesis is that to see a work on the reasoning and to be able to distinguish difficulties due to the concept of implication, there must not be difficulties linked to a mathematical concept. Polyminos' properties are easy to understand and to use. Everyone, even pupils at primary school, as we have already tested, can start a reasoning on polyminos.

Practical organization of the situation
The situation contains two different sessions. The second session allows a work on written proofs that caused difficulties the week before.

Practical organization of the session
Our hypothesis is that a research in groups is necessary for our problems. That allows a confrontation between the various points of view. Furthermore, it stimulates discussions.

Choices for the first problem on polyminos

The tool of coloration
We chose to give the students the property "balanced". Previous experimentations showed that the tool of coloration appears after a very long time. As we had not enough time, we decided to remove the difficulties linked to the modelisation in order to focus our problem on the concept of implication.

The work on implication under the sets point of view
At first sight, one does not know which are the true implications. To answer the problem, one must express conjectures. This is different from a problem of geometry where, most of the time, the result is known before being proved.

Moreover, these implications are true only in one direction whereas in a geometry problem most of implications are equivalences. This compels to distinguish sufficient and necessary conditions.
Our activity on polyminos is focused on the search of counter-examples to refute false implications. We will point out the counter-examples in the transcripts as characteristic of the use of the sets point of view.

Furthermore, the properties of polyminos are not known as are the properties of parallelograms for example. Thus, students can not try to build a proof by writing properties they know, one after another. This prevents the students from using the deductive point of view while it favours the sets one.

Choices for the second problem on polyminos

Second proof
We chose a proof that could seem to be valid whereas the students had proved the implication was false. We will distinguish three types of answers :

- this proof is false since we showed that the implication is false.
- this proof is false since the connexity is not assured.
- this proof is valid but the result is false.

Teacher's opinion
We chose to ask them to answer as a teacher. That allows a discussion on the status of the proof. How to recognize that a speech is a proof ? Which are the criteria which enable them to validate a proof?

CONCLUSION

The analysis of the students' answers is still in progress. However, we can already say that the first problem fulfiled its role, as for the work on the implication since all the groups have worked on necessary and sufficient conditions. Most of the groups studied another implication : Pavable ∪ Non Pavable ⇒ Non pavable ? A group discussed on what is a false implication. Some implications, for example "even ⇒ pavable", are false because there is a counter-example, that is to say they are sometimes false, whereas some implications are always false, for example "odd ⇒ pavable".

Davy: A counter-example, that says that in some cases it is false but it isn't always false whereas here my proof, it is **always** false !

The sets point of view appears many time. Since they do not know any link between the properties they must work with the different classes of polyminos. For example, when they search a counter-example to "balanced ⇒ pavable", they begin with a balanced polymino and they grow it while keeping the number of white boxes equal to the number of black boxes. That it to say they search a non pavable polymino inside the class of balanced polyminos.

These results are to be placed among others. Indeed, this problem forms part of a six hour experimentation on implication and reasoning. Moreover, this experimentation takes sense when one knows that it was preceded by two others, carried out in 1999 and 2000.This problem is, thus, to consider as part of a broader context.

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